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Interpreted systems and Kripke models for multiagent systems from a categorical perspective

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Abstract

Both Kripke models and interpreted systems have been put forward as basic models of multi-agent systems and for reasoning about Knowledge in such systems. This paper enriches previous comparisons of these two forms of semantics by considering categories of models in both cases and then shows that constructions given by Lomuscio and Ryan, extend to give an adjoint equivalence between the two settings. This equivalence is exploited in a discussion of colimits of interpreted systems.

Keywords: Categorical models; Knowledge representation; Epistemic logic; Multiagent systems

1. Introduction

In the study of 'knowledge' in complex systems in artificial intelligence, various logics have played a central role. In multi-agent systems, (MAS), one of the basic logics used is $S5_n$. This is a modal logic first introduced in philosophical logic by Hintikka, [7], and later used in the theory of distributed computing [3]. The best known semantics of modal logics is probably Kripke semantics, based on the theory of (Kripke) Frames, see, for instance, [10] for an up-to-date treatment. The important use of category theory in such sources, in recent work on coalgebraic semantics for modal logics, [1,8,11,18], and, of course, its extensive use within investigations of semantics within theoretical computer science, suggested that it might be a useful additional tool for the theory of multi-agent systems, an area where no categorical treatment was

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yet available. A further benefit of a categorical view should be greater flexibility for extensions to non-deterministic or 'vague' settings.

This is the first of a series of papers for a project: 'Categorical Perspectives on Multiagent Systems', which sets out to see if a categorical approach can shed useful light on this area. As such, it deliberately restricts attention to fairly well known and simple models for MASs so, for instance, the underlying logic will be $S5_n$ and no common knowledge operator is assumed. It seemed better to try out the use of category theory using only an elementary fragment of what is available, for instance, from categorical logic. Of course, as the level of categorical language used is restricted, the descriptive power of that language is limited. To focus the study, this first paper looks at a comparison of Kripke frames with another model for MASs and their inherent logic, namely interpreted systems. These have the advantage that their semantics are somewhat more intuitive and nearer to MAS in their interpretation. This led Lomuscio and Ryan in [12] to ask:

Is one of the approaches more specialised than the other? What is the difference between the two generated logics? Is it possible to use the powerful techniques developed for Kripke models to MAS defined in terms of the more intuitive systems? Is it possible to identify in terms of frames key MAS usually defined in terms of interpreted systems?

In later papers, [13–15], the question of the comparison of these different semantics was explored and the results applied to a number of different problems. The results are not conclusive however although suggestive that the answer should be that the semantics are in some sense 'equipotent'. In this paper the simple notion of morphisms of models is adapted from the categorical treatment of modal logics, and applied to this problem. The result is that the categories of models are seen to be equivalent (apart from some extraneous stuff) and to show that constructions adapted from those given by Lomuscio and Ryan yield a pair of adjoint functors between the two semantic categories. This can be seen as confirming the comparison given by them, but it also raises some additional questions related to the structures that exist in these two categories. As suggested, the constructions and the essential ideas of some of the proofs are in the literature already, but without the simple additional idea of a morphism of models in the two settings the theory ends up seeming incomplete.

As an application of the equivalence results, we examine the existence of colimits in both the Kripke semantic categories and the interpreted system context. The reason for doing this is that it seems possible that many, if not all, interpreted systems can be decomposed as colimits of hypercubes. This might allow analysis of the systems and their associated logics via a modularisation procedure which would have clear advantages for verifying specifications of systems. The category of frames has colimits as is fairly easy to show, but for interpreted systems and sets of global states, the procedure is considerably more complicated. Certain problems that relate to the construction of colimits in these contexts are very closely related to the non-existence of modular formulae to specify those frames that arise from sets of global states.

The level of category theory used has been restricted so as to make the paper more accessible to workers in MASs, where categorical language has not been widely used as yet. This does have a 'downside', of course. Some sections of the paper could be reduced in length by employing a more sophisticated level of categorical machinery. Where this is the case, it has seemed useful to include a few brief technical remarks indicating that deeper level. These remarks can, of course, be ignored by the reader without the specialist categorical knowledge.

The level of category theory used here is kept to a minimum, but the success that subject has had in other related areas of logic and theoretical computer science suggests that a more detailed evaluation of its potential for the study of MAS may be useful.

2. Preliminaries

A positive integer n will be fixed throughout, so there will be n-agents concerned: $A=\{1,2,\dots,n\}$.

Frames. An *equivalence Kripke frame*, *Kripke frame* (or simply *frame*) $F=(W,\sim_1,\dots,\sim_n)$ consists of a set W with, for each $i\in A$, an equivalence relation $\sim_i$ on W. Elements of W are called *worlds* and are denoted w, w', etc. We will write $[w]_i$ for the equivalence class of the element $w\in W$ for the ith equivalence relation, $\sim_i$.

We will not be concentrating much attention on interpretation, at least to start with, but for completeness: an *equivalence Kripke model* $M=(F,\pi)$ is a pair, where F is an equivalence frame and π is an *interpretation* for the atoms of the language being studied (in this case essentially $S5_n$):

$$\pi : W \to \mathcal{P}(P).$$

Alternatively, we can use a *valuation*

$$\pi : P \to \mathcal{P}(W)$$

evaluating the atoms as subsets of the set of possible worlds ($\pi(p)$ is to be thought of as the set of worlds at which p is true.) These viewpoints are dual to each other, and emphasise different viewpoints, but a third, neutral and equivalent viewpoint is to specify the interpretation by a subset

$$R_\pi \subseteq P \times W.$$

The interpretation version gives

$$\pi(w) = \{p \mid (p,w) \in R_\pi\},$$

the valuation version: $\pi(p)=\{w \mid (p,w)\in R_\pi\}$. Various sources we will refer to use one or the other as is most convenient for their context, but translation between them is, of course, routine.

A weak map of frames $f : F \to F'=(W',\sim'_1,\dots,\sim'_n)$ is a function $f : W \to W'$ such that for each i,

$$\text{if } w \sim_i w', \text{ then } f(w) \sim'_i f(w').$$

Frames and the corresponding maps form a category which we will denote $\underline{\mathit{Frames}}_w$. The map f will give a map of models $f:(F,\pi)\to(F',\pi')$ if

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\pi"'] & & W' \arrow[dl, "\pi'"] \\ & \mathcal{P}(P) & \end{tikzcd}$$

commutes.

Note. A weak map is a natural naive notion, but is not well behaved logically. To obtain a better behaved one, it is usual to introduce bounded morphisms (see below). In this paper we will use weak maps when they are sufficient but it is really the bounded morphisms that take the centre stage. We will usually omit 'bounded', but will refer to 'weak maps' as such. The category of frames and weak maps will thus be denoted $\underline{\mathit{Frames}}_w$, whilst that of frames and bounded morphisms by $\underline{\mathit{Frames}}$.

Bounded morphisms (p-morphisms). A Kripke frame is a relational structure and an important class of morphism between two such structures of the same type is that of 'bounded morphisms' or 'p-morphisms. (Goldblatt [6] uses 'bounded' as the existential quantification in the definition is a bounded one. The origins of the term 'p-morphism' are somewhat obscure, but see the discussion of [16, p. 30]. If the equivalence relations of use in the frame are thought of as groupoids, i.e. small categories with all morphisms isomorphisms, then a bounded morphism is a morphism that is a fibration for all of the different groupoid structures involved.) A bounded morphism is also just a functional bisimulation, but we will not be exploring the potential of this fact in this paper.

Definition. A weak map $f:F\to F'$ of frames is *bounded* if for each i, $1\leqslant i\leqslant n$ and $u\in W$, $v'\in W'$, if $f(u)\sim_i' v'$, then there is some $v\in W$ such that $f(v)=v'$ and $u\sim_i v$, i.e.

$$f(u)\sim_i' v' \text{ if and only if } \exists v\in W(f(v)=v' \text{ and } u\sim_i v).$$

Remark. The definition as given above is adapted from [15, Definition 2.4]. but with no assumption of surjectivity. The more formal reprise of the definition is the form given by Goldblatt [6] in the discussion in his Section 4.1. It is, of course, equivalent to the apparently weaker verbal form as f is a weak map.

Global states of interpreted systems. A *set of global states* (SGS) for an interpreted system is a subset S of the product $L_e\times L_1\times\cdots\times L_n$ with each L_e, L_i non-empty. The set L_i represents the local states possible for agent i and L_e the possible states of the environment.

We will restrict attention to the case where L_e has just one element.

An *interpreted system* is a pair (S,π) where

$$\pi:S\to\mathcal{P}(P)$$

as before.

Although $S \subseteq \prod_{i=1}^{n} L_i$, it would seem that the actual sets L_i play only a minor role in the theory. Of more importance will be the decomposition of S into the 'fibres' of the various projection maps

$$p_i : S \to L_i$$

coming from the product structure on $\prod_{i=1}^{n} L_i$. For the moment we will keep the information on the L_i in the notation, but will study more formally later the way it can be discarded.

If $S \subseteq \prod_{i=1}^{n} L_i$, we will write $(S, \underline{L})$ for the SGS, where $\underline{L} = (L_1, \ldots, L_n)$.

A weak map, $\underline{f} : (S, \underline{L}) \to (S', \underline{L'})$, consists of functions $f_i : L_i \to L'_i$ inducing

$$\underline{f} = \prod f_i : \prod L_i \to \prod L'_i,$$

which is required to satisfy

$$\underline{f}(S) \subseteq S'.$$

(Thus if $s = (l_1, \ldots, l_n) \in S$,

$$\underline{f}(s) = (f_1(l_1), \ldots, f_n(l_n)) \in S'.)$$

Remark. (i) In the study of dynamics for interpreted systems, in the related VSK logics [19,20] and in the study of homogeneous broadcast systems [15], the actions or state transformers correspond to weak endomorphisms of the underlying interpreted system.

(ii) The equivalence relations resulting from the projection maps do not positively link states of affairs that the agents can reason about, rather they ascribe properties to the agents, giving a 'birds eye view' of the system. One of the referees of this paper suggested an example from a cryptography context, where one is interested not in deciding "whether or not the attacker knows the key" but "whether or not the attacker has witnessed enough information to be able to deduce the key" (cf. [4]).

(iii) It is perhaps useful to comment on the use of the subsets S of the product $\prod L_i$ in the above definition. This should be seen primarily as resulting from the dynamic construction of the formalism. In examples, the set S is a set of reachable states resulting from protocols, actions or transition functions, depending on the context, from a set of initial states, and, of course, there is no reason to expect all states of the product to be 'reachable'. This is discussed in a little more detail in the paper [17].

Bounded morphisms of global states. We will see later how to translate between frames and systems of global states. Under this translation, the following is the corresponding notion of bounded morphism.

Definition. A weak map $\underline{f} = (f_i)_{i=1}^{n}$, $f_i : L_i \to L'_i$, as above, is said to be a *bounded morphism* if given $w \in S$, $w' \in S'$,

$$\underline{f}(w)_i = w'_i \text{ if and only if } \exists t \in S (\forall j \ \underline{f}(t)_j = w'_j \text{ and } t_i = w_i).$$

A morphism of interpreted systems $\underline{f}:(S,\pi)\to(S',\pi')$ is simply a morphism $\underline{f}:(S,\underline{L})\to(S',\underline{L'})$ for which the diagram

$$\begin{tikzcd} S \arrow[rr, "\underline{f}"] \arrow[dr, "\pi"'] & & S' \arrow[dl, "\pi'"] \\ & \mathcal{P}(P) & \end{tikzcd}$$

commutes.

As with models based on frames, these definitions come in various flavours, as given here, dual ones with 'interpretations' replaced by 'valuations', or neutral using a subset of $P\times S$.

Essential equivalence of morphisms of global states. Two distinct morphisms of SGSs, as defined above, might be different only outside the support of their domain, i.e. on the local states of some agent, they differ at some state, but that state is *not* involved in the *support* S of $(S,\underline{L})$. From the viewpoint of modelling the multiagent systems, the morphisms are clearly 'equivalent'. The definition below captures that notion precisely, but note the need for care with the 'empty SGS'.

Define two maps $\underline{f}_0,\underline{f}_1:(S,\underline{L})\to(S',\underline{L'})$ to be *essentially equivalent* if $\underline{f}_0(s)=\underline{f}_1(s)$ for all $s\in S$, i.e. $\underline{f}_0$ and $\underline{f}_1$ only differ, if at all, away from the "support" S. We will write $\underline{f}_0\simeq\underline{f}_1$ in this case. Two (non-empty) SGSs, $(S,\underline{L})$ and $(S',\underline{L'})$, will be *essentially equivalent* if there are maps

$$\underline{f}:(S,\underline{L})\to(S',\underline{L'}),$$

$$\underline{g}:(S',\underline{L'})\to(S,\underline{L})$$

with

$$\underline{gf}\simeq Id_{(S,\underline{L})},\qquad \underline{fg}\simeq Id_{(S',\underline{L'})}.$$

We would in this case say that $\underline{f}$ is an *essential equivalence*. Of course, this implies that $\underline{g}$ is one as well.

Gloss. This definition needs extending slightly for a technical reason. For any sequence of local states $\underline{L}$, we do have an *empty* SGS based on $\underline{L}$, namely $(\emptyset,\underline{L})$. These empty SGSs clearly should all be essentially equivalent, but the above definition does not make them so since, if any L_i is itself empty, there can be no function to that set from a non-empty one! The extreme case of this is $(\emptyset,\underline{\emptyset})$. There is a unique morphism from $(\emptyset,\underline{\emptyset})$ to any $(S,\underline{L})$, but no morphism back, so with the above definition $(\emptyset,\underline{\emptyset})$ is not equivalent to $(\emptyset,\underline{L})$. We therefore extend the definition by:

In addition any two empty SGSs are deemed to be essentially equivalent with the initial morphisms,

$$(\emptyset,\underline{L})\to(\emptyset,\underline{L}),$$

all declared to be essential equivalences.

Clearly for practically occurring interpreted systems this 'gloss' on the definition will not be needed very often, but it is necessary to avoid awkward exclusions of cases in statements of results later on.

Definition. We will say that $(S,\underline{L})$ is *replete* (or *irredundant*) if for $i=1,2,\dots,n$, $p_i(S) = L_i$; in other words for each $l_i \in L_i$, there is some global state $s \in S$ which involves l_i.

Lemma 1. *Any non-empty* $(S,\underline{L})$ *is essentially equivalent to a replete system of global states.*

Proof. Given any $(S,\underline{L})$, set $\underline{p}(S) = (p_1(S),\dots,p_n(S))$, then $\prod \underline{p}(S) \subseteq \prod \underline{L}$ and S is in bijective correspondence with

$$S' = \{(p_1(s),\dots,p_n(s)):\ s \in S\},$$

since if $s = (l_1,\dots,l_n)$, $p_i(s) = l_i$. Whether or not $S = S'$ is a question that will not concern us, but no harm will come from identifying them so we can consider S as a subset of $\prod \underline{p}(S)$ which gives a SGS, $(S,\underline{p}(S))$, which is replete and comes with natural inclusions $i_k : p_k(S) \to L_k$, $k = 1,\dots,n$, giving a map

$$\underline{i} : (S,\underline{p}(S)) \to (S,\underline{L}).$$

This is claimed to be an essential equivalence. In fact, for each $k = 1,\dots,n$, pick an element $p_k(s_k) \in p_k(S)$ and define $g_k : L_k \to p_k(S)$ by

$$g_k(x) = \begin{cases} x & \text{if } x \in p_k(S), \\ p_k(s_k) & \text{if } x \notin p_k(S). \end{cases}$$

The resulting $\underline{g} = (g_1,\dots,g_n)$ retracts $(S,\underline{L})$ back into $(S,\underline{p}(S))$, moreover by its definition $\underline{gi} \simeq Id_{(S,\underline{p}(S))}$, whilst $\underline{ig}$ differs from $Id_{(S,\underline{L})}$ only away from S, so

$$\underline{ig} \simeq Id_{(S,\underline{L})}. \qquad \square$$

The necessity for 'non-empty' in this result is the reason for our earlier 'gloss' in the definition of essential equivalence.

Hypercube systems. A very natural class of replete interpreted systems is that of hypercubes (cf. [12–15]). Again we will assume L_e is a singleton.

Definition. A *hypercube system* or simply *hypercube* is a system of global states $(S,\underline{L})$ where $S = \prod_{i=1}^{n} L_i$.

The identity intersection property. A useful class of frames is described by the identity intersection property.

Definition. A frame $F = (W, \sim_1,\dots,\sim_n)$ satisfies the *identity intersection property* if $\bigcap_{i=1}^{n} \sim_i$ is the identity equivalence relation.

Any system of global states determines a frame whose ith equivalence relation classes are determined by the fibres of the projection, $p_i : S \to L_i$, and it is clear that if $(S, \underline{L})$ is a system of global states, then the corresponding frame satisfies the identity intersection property. (A proof is given in [15, Lemma 3.1] for hypercubes, but that proof works in more generality.)

We will follow [15] in using the term 'I-frame' for a frame satisfying the identity intersection property.

Categories of global states. Sets of global states together with morphisms between them form a category which will be denoted by $\underline{Glob.States}$. For some purposes a 'quotient' category of $\underline{Glob.States}$ will be more useful. This will have the same objects, but essential equivalence classes of morphisms as its morphisms. (When needed, we will denote this by $[\underline{Glob.States}]$.) The same convention with regard to weak as against bounded morphisms will apply here as for Kripke frames/models, a suffix w will indicate the subcategory with the 'arrows' being the weak maps.

3. An equivalence of categories?

We will initially examine the situation for the categories with weak maps as their arrows and then will check for compatibility with boundedness.

3.1. From global states to frames

Let $(S, \underline{L})$ be a set of global states. Define its associated equivalence frame by

$$\mathcal{F}(S, \underline{L}) = (S, \sim_1, \ldots, \sim_n),$$

where if $\underline{l}, \underline{l}' \in S$,

$$\underline{l} \sim_i \underline{l}' \text{ if and only if } p_i(\underline{l}) = p_i(\underline{l}').$$

Lemma 2. *$\mathcal{F}$ defines a functor*

$$\mathcal{F} : \underline{Glob.States}_w \to \underline{Frames}_w.$$

Proof. This is routine and is left as an exercise. □

Note. Essentially equivalent objects get sent to isomorphic frames.

3.2. … and back again

Given an equivalence frame

$$F = (W, \sim_1, \ldots, \sim_n),$$

we form the sequence

$$\underline{W} = (W_1, \ldots, W_n)$$

with $W_i = W/\sim_i$, the set of equivalence classes of elements of W for the relation, $\sim_i$. There is a 'diagonal' function

$$\Delta : W \to \prod \underline{W}$$

defined by

$$\Delta(w) = ([w]_1, \ldots, [w]_n).$$

Define $\mathcal{G}(F) = (\Delta(W), \underline{W})$.

(N.B. This construction is different from that in [13], but is almost that given by Lomuscio et al. [15], see their Lemma 3.2.)

Lemma 3. *$\mathcal{G}$ defines a functor*

$$\mathcal{G} : \underline{Frames}_w \to \underline{Glob.States}_w.$$

Proof. Just check. □

3.3. Are $\mathcal{F}$ and $\mathcal{G}$ adjoint?

As they link frames and SGSs in a natural way, it is natural to expect that these two functors are adjoints, if not equivalences. Which way around the adjunction should be is fairly clear: $\mathcal{G}$ is defined using a quotienting construction, thus is given by a colimit. This is typical of left adjoints so we will attempt to compare

$$\underline{Frames}_w(F, \mathcal{F}(S, \underline{L}))$$

with

$$\underline{Glob.States}_w(\mathcal{G}(F), (S, \underline{L})).$$

In fact, part of this analysis is already in the literature in another form, cf. [15], but will need interpreting in a categorical form. Here $F = (W, \sim_1, \ldots, \sim_n)$, and $(S, \underline{L})$ is a set of global states.

Suppose $f : W \to \mathcal{F}(S, \underline{L})$ is a weak map of frames, so if $w \sim_i w'$, then $f(w) \sim_i f(w')$. Let $f_i = p_i f : W \to L_i$, then $w \sim_i w'$ implies $f_i(w) = f_i(w')$, so f_i factors canonically as

$$W \xrightarrow{q_i} W/\sim_i \xrightarrow{\overline{f_i}} L_i,$$

where q_i is the natural quotient map $q_i(w) = [w]_i$.

Define

$$\overline{f} = \prod_{i=1}^{n} \overline{f}_i : \prod W_i \to \prod L_i,$$

where, as before, we write W_i for $W/\sim_i$. Thus

$$\overline{f}([w_1]_1, \ldots, [w_n]_n) = (f_1(w_1), \ldots, f_n(w_n)),$$

hence

$$\overline{f}(\Delta(w)) = f(w) \in S,$$

i.e. $\overline{f}(\Delta(W)) \subset S$.

Note that $\overline{f}$ is completely determined by f, so we have defined a function

$$\underline{Frames}_w(F, \mathcal{F}(S, \underline{L})) \to \underline{Glob.States}_w(\mathcal{G}(F), (S, \underline{L})),$$

$$f \to \overline{f}$$

Next, assume given some weak map

$$\underline{g} : \mathcal{G}(F) \to (S, \underline{L}).$$

Thus we are given functions $g_i : W_i \to L_i$, which together define

$$\underline{g} = \prod g_i : \prod W_i \to \prod L_i$$

and for which $\underline{g}(\Delta(W)) \subseteq S$.

Thus for any $w \in W$, setting

$$\widetilde{g}(w) = (g_1[w]_1, \ldots, g_n[w]_n),$$

we have $\widetilde{g}(w) \in S$. This defines a function

$$\widetilde{g} : W \to S$$

and $p_i\widetilde{g}(w) = g_i[w]_i$.

Lemma 4.

$$\widetilde{g} : (W, \sim_1, \ldots, \sim_n) \to (S, \sim'_1, \ldots, \sim'_n)$$

(*where* $s \sim'_i s'$ *if and only if* $p_i(s) = p_i(s')$) *is a weak map of equivalence frames.*

Proof. If $w \sim_i w'$, then $[w]_i = [w']_i$, so $g_i[w]_i = g_i[w']_i$, that is $p_i\widetilde{g}(w) = p_i\widetilde{g}(w')$, as required. □

Lemma 5. (i) *If* $f : W \to \mathcal{F}(S, \underline{L})$, *then* $\widetilde{(\overline{f})} = f$.
(ii) *If* $g : \mathcal{G}(F) \to (S, \underline{L})$, *then* $g = \overline{(\widetilde{g})}$.

Proof. (i) Suppose $w \in W$, $\widetilde{(\overline{f})}(w) = (\overline{f}_1[w]_1, \ldots, \overline{f}_n[w]_n) = (f_1(w), \ldots, f_n(w)) = f(w)$.
(ii) Suppose $([w_1]_1, \ldots, [w_n]_n) \in \prod W_i$,

$$\begin{aligned}\overline{(\widetilde{g})}([w_1]_1, \ldots, [w_n]_n) &= (p_1\widetilde{g}(w_1), \ldots, p_n\widetilde{g}(w_n)) \\ &= (g_1[w_1]_1 \ldots, g_n[w_n]_n) = g([w_1]_1, \ldots, [w_n]_n).\end{aligned}$$

as required. □

To improve notation, let

$$\Phi = \Phi_{F,(S,\underline{L})} : \underline{Frames}_w(F, \mathcal{F}(S, \underline{L})) \to \underline{Glob.States}_w(\mathcal{G}(F), (S, \underline{L})),$$

be defined by

$$\Phi(f) = \overline{f}.$$

We then have that Φ is a natural bijection and $\Phi^{-1}(g) = \widetilde{g}$. (Naturality is easy to check.) Thus:

Theorem 1. *The functor $\mathcal{G}$ is left adjoint to the functor $\mathcal{F}$.*

The importance of many adjointness statements for applications comes from the implications that adjointness has on the transfer of limiting and colimiting constructions—so-called *exactness properties*. We will examine the existence of limits and colimits in these categories later. Another spin-off is the description of adjoints as being solutions to universal properties. This, in turn, reduces to questions of the properties of the unit and counit of the adjunction and it is to these we turn next.

As we have detailed descriptions of Φ and Φ^{-1}, it is relatively easy to examine the unit $\eta_F : F \to \mathcal{F}\mathcal{G}(F)$ and counit $\varepsilon_{(S,\underline{L})} : \mathcal{G}\mathcal{F}(S,\underline{L}) \to (S,\underline{L})$ of the adjunction.

To make this paper more accessible for the non-categorically initiated, let us recall briefly that, in general, if $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ are two functors which are adjoint with a natural isomorphism

$$\Phi : \mathbb{D}(D, FC) \overset{\cong}{\to} \mathbb{C}(GD, C)$$

then we actually know very little about $\mathbb{D}(D, FC)$ for a general arbitrary pair of objects D and C, as there may be <u>no</u> morphisms from D to FC in $\mathbb{D}$. However in one case, namely when $D = FC$, the existence of the identity morphism from FC to itself is guaranteed (by the axioms for categories), so we know

$$id_{FC} \in \mathbb{D}(FC, FC)$$

and thus that $\Phi(id_{FC}) \in \mathbb{C}(GFC, C)$. This morphism $\Phi(id_{FC})$ will be denoted

$$\varepsilon_C : GF(C) \to C.$$

It is called the *counit* of the adjunction and it is standard that given any morphism

$$g : G(D) \to C,$$

there is a unique factorisation of g through ε_C; in fact, $g \in \mathbb{C}(GD, C)$ corresponds to

$$g' = \Phi^{-1}(g) : D \to F(C),$$

which induces (under the application of G)

$$G(g') : G(D) \to GF(C),$$

and $g = \varepsilon_C G(g') : G(D) \to C$. Thus, in a precise sense, this counit (at C) is the universal approximation to C by objects in the image of G. The assignment of the morphism ε_C to the object C gives a natural transformation, which is the *counit of the adjunction*.

Returning to our adjunction between $\mathcal{F}$ and $\mathcal{G}$, let $(S, \underline{L})$ be a set of global states. We have

$$\mathcal{F}(S, \underline{L}) = (S, \sim_1, \ldots, \sim_n),$$

where $s \sim_i s'$ if and only if $p_i(s) = p_i(s')$. Thus for $W = S$ in our earlier description, $W_i = p_i(S)$ and the diagonal function

$$\Delta : S \to \prod p_i(S)$$

sends s to $(p_1(s), \ldots, p_n(s))$. Of course, if $s = (l_1, \ldots, l_n)$ then $p_i(s) = l_i$, so $\Delta(s)$ is 'really' s, but considered as an element of the subset $\prod p_i(S)$ rather than of $\prod L_i$. In fact, $\mathcal{G}\mathcal{F}(S, \underline{L}) = (S, \underline{p}(S))$, the replete system essentially equivalent to $(S, \underline{L})$.

Now the obvious natural morphism from $(S, \underline{p}(S))$ to $(S, \underline{L})$ is that which is the inclusion of the various $p_i(S)$ into L_i. We will see that this is $\varepsilon_{(S, \underline{L})}$.

Consider $id_{\mathcal{F}(S, \underline{L})} : \mathcal{F}(S, \underline{L}) \to \mathcal{F}(S, \underline{L})$, then

$$\Phi(id_{\mathcal{F}(S, \underline{L})}) = \overline{id_{\mathcal{F}(S, \underline{L})}}.$$

The explicit description of the 'bar' construction calculates $p_i id_{\mathcal{F}(S, \underline{L})} : S \to L_i$, which is, of course, p_i itself. It then factors through

$$S \to p_i(S),$$

so $\overline{p_i} : p_i(S) \to L_i$ is the inclusion and $\overline{id_{\mathcal{F}(S, \underline{L})}}$ is the inclusion of $\prod p_i(S)$ into $\prod L_i$. As we have noted that $\Delta(S) = S$ modulo this subset inclusion, we have shown that $\varepsilon_{(S, \underline{L})}$ is as claimed.

As a consequence:

Proposition 1. *If $(S, \underline{L})$ is a replete system, then $\varepsilon_{(S, \underline{L})}$ is an isomorphism.*

Proposition 2 (Lomuscio and Ryan). *If $S = \prod L_i$, so $(S, \underline{L})$ is a hypercube system, then $\varepsilon_{(S, \underline{L})}$ is an isomorphism, (in fact the identity).*

In general, of course, $\varepsilon_{(S, \underline{L})}$ is an essential equivalence.

Proposition 3. *The above adjunction induces one between $\underline{Frames}_w$ and $[\underline{Glob. States}_w]$, making the latter category equivalent to a reflexive subcategory of the category of frames.*

Proof. Although fairly obvious given the previous results (and a simple categorical consequence of the type of adjunction encountered here), we include some idea of the proof for those readers not used to categorical arguments.

For any $(S, \underline{L})$, $\mathcal{F}(S, \underline{L})$ and $\mathcal{F}(S, \underline{p}(S))$ are the same (i.e. equal not merely isomorphic!). For any frame F, $\mathcal{G}(F)$ is a replete system, and by the adjunction between $\mathcal{F}$ and $\mathcal{G}$, any morphism from $\mathcal{G}(F)$ to $(S, \underline{L})$ factors uniquely through $\mathcal{G}\mathcal{F}(S, \underline{L})$. We thus have the following:

Lemma 6.

(i) $\underline{Frames}_w(F, \mathcal{F}(S, \underline{L})) = \underline{Frames}_w(F, \mathcal{F}\mathcal{G}\mathcal{F}(S, \underline{L}))$;

(ii) $$\begin{aligned}\underline{Glob.States}_w(\mathcal{G}(F), (S, \underline{L})) &\cong \underline{Glob.States}_w(\mathcal{G}(F), \mathcal{G}\mathcal{F}(S, \underline{L}))\\ &\cong [\underline{Glob.States}_w](\mathcal{G}(F), (S, \underline{L})).\end{aligned}$$

In fact to shed a little more light on the relationship between $\underline{Glob.States}_w$ and $[\underline{Glob.States}_w]$, consider

$$f_0, f_1 : (S, \underline{L}) \to (S', \underline{L}'),$$

two morphisms in $\underline{Glob.States}_w$, which are essentially equivalent, so $f_0(s) = f_1(s)$ for all $s \in S$. Clearly

$$\mathcal{F}(f_0) = \mathcal{F}(f_1),$$

so $\mathcal{G}\mathcal{F}(f_0) = \mathcal{G}\mathcal{F}(f_1)$. Conversely if $\mathcal{G}\mathcal{F}(f_0) = \mathcal{G}\mathcal{F}(f_1)$ for two weak maps, then

$$f_0 \varepsilon_{(S,\underline{L})} = f_1 \varepsilon_{(S,\underline{L})},$$

so $f_0(s) = f_1(s)$ for all $s \in S$ and $f_0 \simeq f_1$. Thus each equivalence class of maps from $(S, \underline{L})$ to $(S', \underline{L}')$ has a unique representative in

$$\underline{Glob.States}_w(\mathcal{G}\mathcal{F}(S, \underline{L}), \mathcal{G}\mathcal{F}(S', \underline{L}')).$$

Conversely suppose we have a weak map, f from $\mathcal{G}\mathcal{F}(S, \underline{L})$ to $\mathcal{G}\mathcal{F}(S', \underline{L}')$, i.e. from $(S, \underline{p}(S))$ to $(S', \underline{p}'(S'))$, then pick a retraction, *ret* from $(S, \underline{L})$ to $(S, \underline{p}(S))$ by allocating points outside $p_i(S) \subset L_i$ arbitrarily. (The empty case is left to the reader!) Then form the composite

$$(S, \underline{L}) \xrightarrow{ret} (S, \underline{p}(S)) \xrightarrow{f} (S', \underline{p}'(S')) \xrightarrow{\varepsilon_{(S', \underline{p}'(S'))}} (S, \underline{p}(S)).$$

Now apply $\mathcal{G}\mathcal{F}$ to get back to $\overline{f}$ itself. Thus we have

$$[\underline{Glob.States}_w]((S, \underline{L}), (S', \underline{L}')) \cong \underline{Glob.States}_w(\mathcal{G}\mathcal{F}(S, \underline{L}), \mathcal{G}\mathcal{F}(S', \underline{L}')).$$

If $(S, \underline{L})$ is replete, this is isomorphic to $\underline{Glob.States}_w((S, \underline{L}), \mathcal{G}\mathcal{F}(S', \underline{L}'))$, which gives more detail on why (ii) holds.

To sum up there is a natural isomorphism:

(i) $\underline{Frames}_w(F, \mathcal{F}((S, \underline{L}))) \cong [\underline{Glob.States}_w](\mathcal{G}(F), (S, \underline{L}))$;

(ii) the counit $[\varepsilon_{(S,\underline{L})}] : \mathcal{G}\mathcal{F}(S, \underline{L}) \to (S, \underline{L})$ is an isomorphism (since $\varepsilon_{(S,\underline{L})}$ is an essential equivalence).

This means that $[\underline{Glob.States}_w]$ is equivalent, as a category, to a reflexive subcategory of the category of frames. $\square$

This does more than justice to the counit! For fairness we should look at the unit of the adjunction, $\eta : Id_{Frames} \to \mathcal{F}\mathcal{G}$. This should measure the information lost through using $\mathcal{F}\mathcal{G}(F)$ rather than F itself.

Let $F = (W, \{\sim_i\}_{i=1}^n)$, then $\mathcal{F}\mathcal{G}(F) = (\Delta(W), \{\sim_i'\}_{i=1}^n)$, where

$$\Delta : W \to \prod_{i=1}^n W_i \quad \text{with } W_i = W/{\sim_i},$$

and $\sim_i'$ is defined using the projections. The morphism $\eta_F : F \to \mathcal{F}\mathcal{G}(F)$ sends w to the string $([w]_1, \ldots, [w]_n)$. This is a morphism, since if

$$w \sim_i w' \quad \text{then } p_i\Delta(w) = [w]_i = [w']_i = p_i\Delta(w').$$

The properties of η relate well to the results found by Lomuscio and Ryan in [13].

Lemma 7. *The unit*, η_F, *is an isomorphism if and only if* $\bigcap_{i=1}^n \sim_i = id_W$.

Proof. If $\Delta(w) = \Delta(w')$, then $w \sim_i w'$ for all i so η_F is one–one if and only if the intersection of the equivalence relations is discrete. As Δ is always onto, this completes the proof. □

In fact, this lemma gives us the internal description of the image of the category $[\underline{Glob.States}_w]$ in $\underline{Frames}_w$ and thus of the reflection, namely that image is the full subcategory of I-frames.

Proposition 4. *The full subcategory*, I-$\underline{Frames}_w$, *of* $\underline{Frames}_w$ *determined by those Kripke equivalence frames that satisfy the identity intersection property*, *is a reflexive subcategory with reflection* $\mathcal{L} = \mathcal{F}\mathcal{G}$. *This subcategory is equivalent to that of global states with essential equivalence classes of weak maps.*

Proof. This is an easy consequence, categorically, of the properties of the adjoint equivalence already discussed in detail. For instance, $\mathcal{L}$ is idempotent, $\mathcal{L}^2 = \mathcal{L}$, since $[\varepsilon_{(S,\underline{L})}]$ is an isomorphism, so

$$\mathcal{L}^2 = \mathcal{F}\mathcal{G}\mathcal{F}\mathcal{G} \xrightarrow{\mathcal{F}(\varepsilon_{\mathcal{G}})} \mathcal{F}\mathcal{G} = \mathcal{L}$$

is one as well.

One can also approach it directly using the intersection of the equivalence relations. Define

$$\sim = \bigcap_{i=1}^n \sim_i;$$

for a frame $F = (W, \sim_1, \ldots, \sim_n)$, form $F/{\sim}$ with set of worlds $W/{\sim}$ and with the induced equivalence relations, which will be denoted $\approx_i$; thus if $[w]$ denotes the $\sim$-equivalence class of w,

$$[w] \approx_i [w'] \text{ if and only if } w \sim_i w'.$$

Of course, this is well defined and the quotienting map

$$g_F : F \to F/\sim$$

is a weak map of frames, by construction.

Lemma 8. (i) *For any F, the quotient frame $F/\sim$ is an I-frame.*

(ii) *If $f : F \to F'$ is any weak map with F' an I-frame, then f factors uniquely via q_F:*

$$\begin{tikzcd}
F \arrow[rr, "f"] \arrow[dr, "q_F"'] & & F' \\
& F/\sim \arrow[ur, "\exists! f'"'] &
\end{tikzcd}$$

(iii) *The map q_F is a bounded morphism.*

(iv) *For any frame, F, $(F/\sim) \simeq \mathcal{FG}(F)$, and, up to this isomorphism, $\eta_F = q_F$.*

Proof. Most of this is routine to check, so we will just illustrate the proof of (iii).

(iii) Suppose $u \in W$, $v' \in W/\sim$ with $q_F \approx_i v'$, then there is some $w \in W$ with $[w] = v'$ and $[u] \approx_i [w]$, i.e. $u \sim_i w$, but then this w is exactly what is needed for the definition of bounded morphism. (It is worth remarking that this situation is special as *the same* 'lift' of v' can be used for all i. The definition of bounded morphism only assumes a lift exists for each i.) $\square$

This lemma also completes the proof of the proposition. $\square$

The lemma, in fact, tells us more: namely that q_F (that is η_F) is a bounded morphism that is split, i.e. there is a splitting

$$F/\sim \xrightarrow{s} F$$

obtained by choosing a representative for each $\sim$-equivalence class (using the Axiom of Choice if the set W is infinite). That such a setwise splitting defines one at the relational/frame level is easily seen, but the splitting will rarely be bounded, only being so when F is already an I-frame in which case q_F is an isomorphism.

This almost completes the comparison of the underlying structures for interpreted systems and Kripke models for basic $S5_n$-systems. We have seen that

(i) sets of global states may have redundancy, i.e. may not be replete;

(ii) Kripke frames may have worlds that are not distinguishable by any of the equivalence relations, i.e. in the multi-agent interpretation, are not distinguishable by any agents, so for the system, they are equivalent worlds.

If one eliminates redundancy and the existence of indistinguishable worlds, the two settings are equivalent. No multi-agent system can see states that are not in S, even though individual agents may 'see' the image in their state space of such a state and, as was just said, no multi-agent system can distinguish indistinguishable worlds! This also implies they cannot test if a frame is an I-frame (which is the interpretation of the result that the I-property is not describable in terms of modal operators).

Categorical aside. The above argument is, of course, just a simple instance of a general categorical fact. The adjunction of Theorem 1 induces an idempotent monad on $\underline{Frames}_w$ and an idempotent comonad on $\underline{Glob.States}_w$. The monad

$$\mathcal{L} : \underline{Frames}_w \to \underline{Frames}_w$$

send a frame to its quotient with respect to the largest bisimulation, and general facts about idempotent monads yield the equivalences that have been noted. (Useful references for idempotent monads are Borceux [2], vol. II; Johnstone [9, Chap. IV]). The generality of the categorical approach is also highly relevant if one wishes to consider weakening from $S5_n$-based logics to ones with order theoretic semantics.

The above comparison does not address the question of models, only the underlying relational structures. We will look shortly at the geometric models. Before doing that it will be useful to consider what the adjoint equivalence does to bounded morphisms.

First observe that the composite of bounded morphisms is bounded, both for frames and for sets of global states. Because of this we can form a category from global states (resp. frames) and bounded morphisms only. These will be denoted $\underline{Glob.States}$ (resp. $\underline{Frames}$).

Proposition 5. (i) *If $f : F \to F'$ is a bounded morphism of frames, then $\mathcal{G}(f)$ is a bounded morphism of global states.*

(ii) *If $\underline{G} : (S, \underline{L}) \to (S', \underline{L}')$ is a bounded morphism of global states, then $\mathcal{F}(\underline{g})$ is a bounded morphism of frames.*

(iii) *The unit and counit of the $\mathcal{G}$, $\mathcal{F}$ adjunction are bounded morphisms.*

Proof. Condition (i) is simply a matter of notation: the given condition is that

$$\mathcal{G}(f)(\Delta(\omega))_i = \Delta(\omega')_i,$$

but this is a convoluted way of noting that

$$f(\omega) \sim_i \omega'.$$

Applying the fact that f is bounded gives a $v \in W$ and $t = \Delta(v)$ does the job for the conclusion.

Condition (ii) is even more direct and is left to the reader.

Condition (iii) can be approached in several different ways. It has already been noted for η_F and $\varepsilon_{(S,\underline{L})}$ is an essential equivalence. Any essential equivalence is bounded, as is easily checked. $\square$

Theorem 2. *The functors $\mathcal{F}$ and $\mathcal{G}$ induce an adjoint pair, (which will also be denoted $\mathcal{F}$, and $\mathcal{G}$) between $\underline{Frames}$ and $\underline{Glob.States}$.*

Proof. We start by proving adjointness, i.e. that Φ induces another Φ,

$$\Phi : \underline{Frames}(F, \mathcal{F}(S, \underline{L})) \cong \underline{Glob.States}(\mathcal{G}(F), (S, \underline{L})).$$

This follows from the previous proposition:
if $f : F \to \mathcal{F}(S, \underline{L})$, then $\Phi(f) = \bar{f}$ as described earlier. It is standard that

$$\Phi(f) = \varepsilon_{(S,\underline{L})}.\mathcal{G}(f)$$

and if f is bounded then by the proposition, so is $\mathcal{G}(f)$. The counit is naturally bounded and composites of bounded morphisms are bounded, so $\Phi(f)$ is bounded.

The verification that, if $g : \mathcal{G}(F) \to (S, \underline{L})$ is bounded, then so is $\tilde{g} = \Phi^{-1}(g) = \mathcal{F}(g).\eta_F$, is almost identical.

Of course, it is also easy to verify this directly. □

Of course, restricting to I-*Frames* and passing to [*Glob.States*] gives an equivalence of categories. This effectively answers the questions posed by Lomuscio and Ryan [12]. The only possible extra power of Kripke frames would be if they could express the difference between ordinary frames and I-frames, but in [15], a neat example is given that shows that no modal formula can correspond to property I. Thus these two forms of geometric semantics have equivalent expressive power.

4. Compatibility with the models

We have not yet considered the compatibility of the functors with the model structures, in other words we have not considered whether they preserve the logic.

Kripke models. Earlier we defined $f : (F, \pi) \to (F', \pi')$ to be a map of models if

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\pi"'] & & W' \arrow[dl, "\pi'"] \\ & \mathcal{P}(P) & \end{tikzcd}$$

commutes. Recalling the notion of satisfaction of a formula ϕ in a world w of $M = (F, \pi)$ and the corresponding notion of validity of ϕ, we note

$$M, w \models p \quad \text{if } p \in \pi(w),$$

$$M, w \models \neg\phi \quad \text{if } M, w \not\models \phi,$$

$$M, w \models \phi \wedge \psi \quad \text{if } M, w \models \phi \text{ and } M, w \models \psi,$$

$$M, w \models \Box_i \psi \quad \text{if for each } w' \in W, \text{ with } w \sim_i w', \text{ we have } M, w' \models \phi.$$

A formula ϕ is *valid* on a model M if $M, w \models \phi$ for every $\omega \in W$.

(We will sometimes use $F, w \models_\pi \phi$ as an alternative notation for $(F, \pi), w \models \phi$, when this makes the formula easier to read or when it is more convenient for the context.)

The notion of satisfaction for an interpreted system, (S,π), suppressing the local states L_i in the notation, is given, for instance, in [5]. This is explicitly in terms of the associated Kripke model, that is what we have denoted $\mathcal{F}(S,\underline{L})$. Explicitly, in our notation, if $s \in S$ is a state of the system

$$S, s \models_\pi \phi \text{ if and only if } \mathcal{F}(S,\underline{L}), s \models_\pi \phi,$$

the latter in the sense already discussed.

Proposition 6. *Given $M=(F,\pi)$ a Kripke model with F an I-*frame, *and $w \in W$, a world,*

$$M, w \models \phi \textit{ if and only if } (\mathcal{G}(F), \pi\eta_F^{-1}), \Delta(w) \models \phi.$$

Proof. This follows easily from the fact that

$$\eta_F : F \to \mathcal{F}\mathcal{G}(F)$$

is an isomorphism if F is an I-frame. As the set of states/worlds of $\mathcal{G}(F)$ and of $\mathcal{F}\mathcal{G}(F)$ are the same, the interpretation $\pi\eta_F^{-1}$ does make sense. $\square$

If F is an ordinary frame, i.e. not necessarily an I-frame, then this result does not seem to have an analogue in general. It does, however, provided that the interpretation $\pi : W \to \mathcal{P}(P)$ satisfies an obvious compatibility condition.

We will say that the interpretation π is *compatible* with the frame structure on W if

$$w_1 \sim w_2 \text{ implies } \pi(w_1) = \pi(w_2).$$

Proposition 7. *If $F=(W,\sim_1,\ldots,\sim_n)$ is a Kripke frame and $M=(F,\pi)$ with π compatible with the frame structure, then there is an interpretation $\pi' : \mathcal{G}(F) \to \mathcal{P}(P)$ such that for any formula ϕ and world w,*

$$M, w \models \phi \textit{ if and only if } (\mathcal{G}(F), \pi'), \Delta(w) \models \phi.$$

Proof. If $\underline{x} = ([x]_1,\ldots,[x]_n) \in \Delta(W)$, define $\pi'(\underline{x}) = \pi(x)$. This is well defined. The proof now proceeds in the obvious way by an induction. We indicate the first step only.

Suppose $M, w \models p$ then $p \in \pi(w) = \pi'(\underline{w})$, so $(\mathcal{F}\mathcal{G}(F), \pi'), \eta_F(w) \models p$, i.e. $(\mathcal{G}(F), \pi')$, $\Delta(w) \models p$. The argument is reversible and the induction is now standard. $\square$

Before interpreting these results for the two types of models, it is important to connect up the notion of morphism of models with other notions from the modal logic literature. We use Kracht [10] as a source, but note that he uses valuations ${}_L\pi : P \to \mathcal{P}(W)$ rather than interpretations $\pi : W \to \mathcal{P}(P)$, thus the forms of definitions will look somewhat different. Also we have given them in a form for equivalence frames, not the more general form used in [10].

Suppose $f:F\to F'$ is a morphism of frames and $\pi:W\to\mathcal{P}(P)$ is an interpretation of the worlds of F. We say f is *admissible* for π if for any $w_1,w_2\in W$

$$f(w_1)=f(w_2) \text{ implies } \pi(w_1)=\pi(w_2).$$

Comparison with the above discussion on 'compatibility' shows that the frame map

$$q_F:F\to F/{\sim}$$

is admissible for π if and only if π is compatible with the frame structure.

Another example is if

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\pi"'] & & W' \arrow[dl, "\pi'"] \\ & \mathcal{P}(P) & \end{tikzcd}$$

commutes for some π', then f is admissible for π.

This raises the interesting inverse question as to whether or not given

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\pi"'] & & W' \\ & \mathcal{P}(P) & \end{tikzcd}$$

there is a suitable π'. We split the question in two:

(i) is there an optimal π' such that

$$\pi\leqslant\pi' f$$

(where $\pi\leqslant\pi' f$ is to be interpreted as

$$\text{for all } w\in W,\ \pi(w)\subseteq\pi' f(w)).$$

We will call such an optimal π' a *minimal right lax extension* of π along f.

Specifically we want a right lax extension of π along f to be an interpretation $\pi^f:W'\to\mathcal{P}(P)$ such that:

(a) $\pi\leqslant\pi^f f$;
and
(b) if $\pi':W'\to\mathcal{P}(P)$ is an interpretation with $\pi\leqslant\pi' f$, then $\pi^f\leqslant\pi'$.

(ii) Dually, we ask if there is a *maximal left lax extension*, i.e. an interpretation $\pi_f:W'\to\mathcal{P}(P)$ such that

(a) $\pi_f f\leqslant\pi$;
and
(b) if $\pi':W'\to\mathcal{P}(P)$ is an interpretation with $\pi' f\leqslant\pi$, then $\pi'\leqslant\pi^f$.

The existence of both maximal and minimal extensions is assured.

Lemma 9. *Given $f: F \to F'$ and an interpretation $\pi: W \to \mathcal{P}(P)$ then for $w' \in W'$*

$$\pi_f(w') = \bigcup\{\pi(w) | f(w) = w'\}$$

and

$$\pi^f(w') = \bigcap\{\pi(w) | f(w) = w'\}.$$

Proof. The proof is easy from the definitions. □

Proposition 8. *If f is admissible for π then*

$$\pi_f(w') = \pi^f(w')$$

for all $w' \in f(W)$.

Proof. If $f(w_1) = f(w_2) = w'$, then $\pi(w_1) = \pi(w_2)$ so the family of sets of which one takes the union/intersection consists of one set only! □

The importance of this is that if f is admissible for π, then

$$\pi_f(w') = \pi(w) \quad \text{if } w' = f(w)$$

and this is well defined on the image of f. Of course, $\pi_f(w')$ is *globally* different from $\pi^f(w')$ in general, i.e. for w' outside the image of f, since if $\{\pi(w) | f(w) = w'\}$ is the empty family, then its union is $\emptyset$ and its intersection is P itself as the relations

$$\pi^f \leqslant \pi' \leqslant \pi_f$$

imply for any π' completing the triangle. (Of course, surjectivity of f would give $\pi_f = \pi^f$ globally on W'.)

The above implies that the 'internal' condition on (f, π) that is admissibility, is equivalent to there being a morphism of models

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\pi"'] & & W' \arrow[dl, "\pi'"] \\ & \mathcal{P}(P) & \end{tikzcd}$$

where different π' can, of course, differ outside the image of f.

Proposition 9. *If $f: (F, \pi) \to (F', \pi')$ is a morphism of Kripke models and $w \in W$ is a possible world of F, then for a formula ϕ,*

$$(F, \pi), w \models \phi \textit{ implies } (F', \pi'), f(w) \models \phi.$$

Proof. This is a standard result and is, for instance, a consequence of the proof of Proposition 2.4.2, [10, p. 66]. □

Before we can compare interpreted systems with Kripke models, we note the corresponding idea of 'admissibility' for interpreted systems.

A morphism $\underline{f}:(S,\underline{L})\to(S',\underline{L'})$ of global states will be said to be *admissible* for an interpretation

$$\pi : S \to \mathcal{P}(P)$$

if $\underline{f}(s_1)=\underline{f}(s_2)$ implies $\pi(s_1)=\pi(s_2)$.

Remarks (All more or less obvious). (i) If $\underline{f}:((S,\underline{L}),\pi)\to((S',\underline{L'}),\pi')$ is a morphism of interpreted states, then f is admissible for π.

(ii) If $\underline{f}:(S,\underline{L})\to(S',\underline{L'})$ is admissible for π, then the induced

$$\mathcal{F}(f):\mathcal{F}(S,\underline{L}) \to \mathcal{F}(S',\underline{L'})$$

is admissible for the (induced) interpretation $\pi:S\to\mathcal{P}(P)$ (i.e. considered as part of the Kripke model $((S,\sim_1,\dots,\sim_n),\pi)$ rather than as an interpreted system).

(iii) There are analogues of the lax extension results for pairs (f,π) in the context of interpreted systems. Within the multiagent setting, a morphism $\underline{f}=\{f_i\}_{i=1}^n$ translates the local world's view of agent i in the first system into the corresponding local view of the corresponding agent i in the second:

$$f_i:L_i \to L'_i.$$

For $s_1,s_2\in S$, if for some i, $f_i(s_1)=f_i(s_2)$, but the interpretation π distinguishes the states s_1 and s_2 (so $\pi(s_1)\neq\pi(s_2)$) then f_i will be destroying the potential knowledge of (S,π) as an interpreted system and $\underline{f}$ will not be admissible for π.

For two interpretations $\pi,\pi':S\to\mathcal{P}(P)$, $\pi\leqslant\pi'$ if for all $s\in S$, $\pi(s)\subseteq\pi'(s)$.

If (S,π) is an interpreted system and $s\in S$ is a global state, it is natural to think of the set

$$Th((S,\pi),s)=\{\phi \mid (S,\pi),s\models\phi\}$$

as the theory of that interpreted system at state s. This is a theory in the usual sense of modal logic, (cf. [10]). Of course, if $\mathcal{M}=(\mathcal{F}(S,\underline{L}),\pi)$ is the corresponding Kripke model, it is just the theory of $\mathcal{M}$ at s, in the usual sense, so we are not really doing anything new here:

If $\pi\leqslant\pi'$ then $Th((S,\pi),s)\subseteq Th((S,\pi'),s)$.

Given this, the interpretation of the lax extensions for interpreted systems is clearer. Interpreting $\underline{f}$ as a translation of one set of n-agents collective view to another set of n-agents, π' satisfies

$$\pi' f \geqslant \pi$$

if the theory $Th((S,\pi' f),s)\supseteq Th((S,\pi),s)$. However for a $p\in P$ and a global state, $s\in S$,

$$\begin{aligned}(S,\pi' f),s\models p \text{ if and only if } & p\in\pi' f(s)\\ \text{if and only if } & (S',\pi'),f(s)\models p,\end{aligned}$$

so

$$Th((S',\pi'),f(s)) = Th((S,\pi),s).$$

Putting this together gives: if π' is a right lax extension, then

$$Th((S,\pi),s) \subseteq Th((S',\pi'),f(s)).$$

The minimal right lax extension of π along f, thus gets as few extra propositions valid as possible, similarly for the maximal left extension of π along f.

Returning to our discussion of adjointness, we can summarise what the situation is: we have an adjoint equivalence

$$\mathcal{F} : \underline{Glob.States} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \underline{Frames} : \ \mathcal{G}$$

Is there an induced equivalence:

$$\mathcal{F} : \underline{Int.Systems} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \underline{Kripke.Models} : \mathcal{G}\ ?$$

There is one problem but that has already been examined earlier, although not explicitly in this context. We have for a frame F, the unit

$$\eta_F : F \to \mathcal{F}\mathcal{G}(F)$$

and we saw this is the same as

$$q_F : F \to F/{\sim},$$

but that if π is an interpretation, $\pi : W \to \mathcal{P}(P)$, it need not be compatible for the frame structure. It might happen that:

$$w \sim w' \quad \text{but } \pi(w) \neq \pi(w'),$$

thus there would be some p which is in one of $\pi(w)$, $\pi(w')$, but not the other. Without loss of generality, assume $p \in \pi(w)$ but $p \notin \pi(w')$. Then although each agent i can observe p is true at w in the Kripke model (F,π), no agent <u>knows</u> that p is true at w since for no agent is $\Box_i p$ true. This phenomenon cannot occur with interpreted systems (since $w \sim w'$ if and only if $w = w'$). A related discussion can be found in [13].

Restricting to Kripke models (F,π) with π compatible with the frame structure then we get a category $\underline{Comp.Models}$ and a pair of adjoint functors

$$\mathcal{F} : \underline{Int.Systems} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \underline{Comp.Models} : \mathcal{G}$$

and these define an equivalence of categories.

5. Colimits of frames and of interpreted systems

Coproducts generalise disjoint unions; colimits generalise unions. Coproducts are already current tools in modal logic and, in fact, in the emerging theory of covarieties of coalgebras, they are an essential part of the structure needed for a dual version of Birkhoff's theorem (see [1,8,11]), and thus for the coalgebraic semantics of modal logics.

The potential for a use of coproducts and more general colimits within a multiagent theory is considerable. Here we might consider decomposition theories for interpreted systems, e.g. as colimits of simpler systems such as hypercubes; as Kripke frames generalise labelled transition systems, decomposition of frames as colimits may help to 'modularise' the systems involved, allowing greater ease of specification and later checking of implementation of any programs modelling the logical behaviour.

That general colimits *do* exist in these categories is not in doubt. Kripke frames and models form categories of coalgebras in the sense, say, of Kurz [11] or Rutten [18] whilst [$\underline{Glob.States}$] is equivalent to a coreflexive subcategory of that category and colimits are preserved by coreflexions. In this section, we will extract explicit construction of such colimits. It is hoped that this will indicate to the less 'categorically initiated' some of the potential of these notions, the simplicity of their constructions and their relationship and relevance to important questions in the modelling of multiagent systems.

Coproducts of Kripke frames are easy to construct, see for example [10, p. 67]. We summarise the construction:

Suppose $F_i = (W_i, \sim_1, \dots, \sim_n)$, $i \in I$, is a family of Kripke frames. Form a new frame

$$\coprod_{i\in I} F_i$$

with set of possible worlds $\bigsqcup_{i\in I} W_i \times \{i\}$, i.e. the disjoint union of the W_i and with, for $k = 1, 2, \dots, n$, $\sim_k$ defined by

$$(\omega, i) \sim_k (\omega', j) \text{ if and only if } \omega \sim_k \omega' \text{ and } i = j,$$

so no equivalence class can contain elements from different components.

If $\pi_i : W_i \to \mathcal{P}(P)$ are interpretations, then the coproduct property of disjoint unions of sets gives an interpretation

$$\pi : \coprod_{i\in I} W_i \to \mathcal{P}(P)$$

given by $\pi(w, i) = \pi_i(w)$.

Proposition 10. (i) *The canonical mappings*

$$e_j : F_j \to \coprod_{i\in I} F_i$$

are bounded morphisms of frames, (*similarly for models*).

(ii) *The frame* $\coprod_{i\in I} F_i$ *is the coproduct of the* F_i, *so given any* (*bounded*) *morphisms*

$$f_i : F_i \to G \quad (\textit{resp. } f_i : (F_i, \pi_i) \to (G, \pi'))$$

of Kripke frames (*resp. models*), *in the corresponding categories*, *there is a canonical* (*unique*) $f : \coprod F_i \to G$ *such that* $f_i = f e_i$.

(iii) $Th(\coprod F_i) = \bigcap Th(F_i)$.

Proof. See [10, pp. 67–68]. □

Suppose $(S_i, \underline{L}_i)$, $i \in I$, is a family of sets of global states with $\underline{L}_i = (L_{i,1}, \ldots, L_{i,n})$ and $S_i \subseteq \prod_{j=1}^{n} L_{i,j}$.

Define for $k = 1, \ldots, n$, $(\coprod L_i)_k = \bigcup L_{i,k} \times \{i\}$, the disjoint union of the $L_{i,k}$ for $i \in I$.

Define $\coprod S_i$ by $\bigcup S_i \times \{i\}$. The various maps from $L_{i,k}$ to $\coprod L_{i,k}$ define a mapping

$$S_i \subseteq \prod L_{i,k} \to \prod_k \coprod_i L_{i,k}$$

and thus by the coproduct property, there is a unique

$$\coprod S_i \to \prod_k \coprod_i L_{i,k}.$$

It is worth noting that this product $\prod_k \coprod_i L_{i,k}$ is usually very different from $\coprod_i \prod_k L_{i,k}$, thus, for instance, if $A = \{1, 2\}$, the former includes parts such as $L_{1,k} \times L_{2,l}$ for $k \neq l$, but no such terms occur in the other expression. In particular even if $(S_1, \underline{L}_1)$ and $(S_2, \underline{L}_2)$ are hypercube systems, their coproduct will not, in general, be one.

Again if interpretations are specified, the coproduct of the interpreted systems is defined with the above as underlying set of global states and with the induced interpretation.

An advantage of a categorical approach is that to check that coproducts of, say, interpreted systems give one coproducts of the associated Kripke equivalence models is more or less immediate. Of course, a direct proof can also be given. Only slightly more difficult is that the functor $\mathcal{G}$ from frames to global states also preserves coproducts. (Categorically, it is a left adjoint so will preserve colimits.)

It is standard category theory that general existence of colimits follows from existence of coproducts and coequalisers. We will restrict our discussion to handle the case of colimits in the categories having bounded morphisms as this is more important for applications. The data for a coequaliser diagram is given in the form

$$F_1 \underset{b}{\overset{a}{\rightrightarrows}} F_2$$

where a, b are morphisms of frames. The aim is to form a 'universal' quotient

$$F_2 \xrightarrow{c} F_3$$

having the property that $ca = cb$. At the level of underlying sets, the obvious thing to do is to form the smallest equivalence relation on W_2, the set of worlds of F_2, containing the pairs (aw_1, bw_1) for all $w_1 \in W_1$, with the obvious notation. This equivalence relation is a net, (cf. Kracht ([10], p. 66)), i.e. writing it as R, if $x, x' \in W_2$ and xRx', and in addition, $x \sim_k, y$, then there is a $y' \in W_2$, with $y \sim_k y'$ and $x'Ry'$. The proof uses a simple induction on the length of the zig-zag of basic pairs, $a(w)Rb(w)$ or $b(w)Ra(w)$, linking x and x', together with repeated use of the boundedness of a and b.

This implies that, if

$$W_3 := W_2/R$$

with $c : W_2 \to W_3$ given by $c(x) = [x]_R$, then there is an induced equivalence relation $\sim_k$ on W_3 for each $k \in A$, defined by

$$[x]_R \sim_k [y]_R$$

if and only if there are elements $x' \in [x]_R$, $y' \in [y]_R$ with $x' \sim_k y'$ in W_2.

The mapping $c : W_2 \to W_3$ underlies a bounded morphism of frames

$$c : F_2 \to F_3 := (W_3, \sim_1, \ldots, \sim_n).$$

It is then easy to check that c is the coequaliser of a and b.

The case of weak maps is easier, but far less useful for applications, so will be omitted.

Earlier we mentioned that the class of I-frames was not modally definable (see [13]). The proof used that class was not closed under bounded homomorphic images. From a categorical viewpoint, this says that the category of I-frames is not a covariety in the category of all Kripke equivalence frames with bounded morphisms. The subcategory I-$\underline{\mathit{Frames}}$ of $\underline{\mathit{Frames}}$ is a reflexive subcategory (if F is a general frame and F' an I-frame, then any (bounded) frame morphism $f : F \to F'$ factors through $F/\sim$). The inclusion is right adjoint to the functor

$$\mathcal{L} : \underline{\mathit{Frames}} \to I\text{-}\underline{\mathit{Frames}}$$

with $\mathcal{L} = \mathcal{F}\mathcal{G}$, $\mathcal{L}(F) = F/\sim$, but right adjoints do not, in general, preserve colimits, so does this one preserve our coequalisers? The answer is no. This corresponds closely to the non-modal definability of I-frames, but is also of importance when considering the question of coequalisers for SGSs. To emphasise the close links, we will give an example of the colimit of a pair of bounded morphisms of I-frames whose coequaliser in $\underline{\mathit{Frames}}$ is not an I-frame. (In fact, this is an adaptation of the neat example used by Lomuscio and Ryan [13] to show non-modal definability of I-frames.)

Example. Let $W = \{w_1, w_2, w_3, w_4\}$ with $\sim_1$ generated by the basic pairs

$$w_1 \sim_1 w_3 \qquad w_2 \sim_1 w_4$$

and $\sim_2$ by basic pairs

$$w_1 \sim_2 w_2 \qquad w_3 \sim_2 w_4,$$

so the frame $F = (W, \sim_1, \sim_2)$ looks like

$$\begin{tikzcd} w_1 \arrow[r, dash, "2"] \arrow[d, dash, "1"'] & w_2 \arrow[d, dash, "1"] \\ w_3 \arrow[r, dash, "2"'] & w_4 \end{tikzcd}$$

It is clear that F is an I-frame.

Let $a : F \to F$ be the identity morphism and $b : F \to F$, the obvious automorphism given by

$$b(w_1) = w_4,$$
$$b(w_2) = w_3,$$
$$b(w_3) = w_2,$$
$$b(w_4) = w_1.$$

We want to consider the coequaliser of the pair

$$F \overset{a}{\underset{b}{\rightrightarrows}} F.$$

Our recipe for its construction gives:

- Define R: This is simply generated by

  $$w_1 R w_4 \qquad w_2 R w_3$$

  i.e. $awRbw$ for each $w \in W$.
- Form $W_3 = W/R$: So W_3 has two elements, which we will denote by

  $$x = \{w_1, w_4\},$$

  and

  $$y = \{w_2, w_3\}.$$

- Define the relational structure on W_3: Here the recipe gives

  $$x \sim_1 y \quad \text{since } w_1 \sim_1 w_3$$

  and

  $$x \sim_2 y \quad \text{since } w_1 \sim_2 w_2.$$

Thus the coequaliser, F_3, in $\underline{Frames}_b$ is given by the graph

$$x \overset{1,2}{\text{———}} y$$

and, of course, this is not an I-frame as x and y are indistinguishable.

The coequaliser of this diagram within I-frames is $\mathcal{L}(F_3)$, since $\mathcal{L}$, being a left adjoint preserves coequalisers. This collapses F_3 to a single world with the identity equivalence relation on it.

Coequalisers of morphisms of SGSs are more of a problem to describe explicitly. In $\underline{Glob.States}_w$, they exist since one just takes the coequalisers of the component morphisms as the sets of local states of each agent, then takes the product of these and within that looks at the image set of global states. More precisely, if

$$\underline{a}, \underline{b} : (S_1, \underline{L}_1) \to (S_2, \underline{L}_2),$$

then for each agent, i, we form the coequaliser $L_{3,i}$ of

$$a_i, b_i : L_{1,i} \to L_{2,i}.$$

This comes with an induced function

$$c_i : L_{2,i} \to L_{3,i}$$

and we let $S_3 = \underline{c}(S_2)$ be the image subset of $\prod \underline{L}_3$ under the function

$$\underline{c} = (c_i)_{i=1}^n.$$

As an example of this consider the SGSs corresponding to the above example under the functor, $\mathcal{G}$. Here $W_1 = \{\{w_1, w_3\}, \{w_2, w_4\}\}$, and to simplify the notation we will write $w_{1,3} = \{w_1, w_3\}$, etc. Thus W_1, which is $W/\sim_1$, will be $\{w_{1,3}, w_{2,4}\}$, and similarly

$$W_2 := W/\sim_2 = \{w_{1,2}, w_{3,4}\}.$$

The function

$$\Delta : W \to W_1 \times W_2$$

gives

$$\begin{aligned}
\Delta(w_1) &= (w_{1,3}, w_{1,2}),\\
\Delta(w_2) &= (w_{2,4}, w_{1,2}),\\
\Delta(w_3) &= (w_{1,3}, w_{3,4}),\\
\Delta(w_4) &= (w_{2,4}, w_{3,4}).
\end{aligned}$$

As to the morphisms, $\underline{a}$ is the identity and $\underline{b}$ has coordinate functions

$$b_1(w_{1,3}) = w_{2,4}, \qquad b_1(w_{2,4}) = w_{1,3},$$

$$b_2(w_{1,2}) = w_{3,4}, \qquad b_2(w_{3,4}) = w_{1,2},$$

that is the obvious permutations of W_1 and W_2.

$$\underline{L}_1 = \underline{L}_2 = (W_1, W_2).$$

The set of local states of the coequalisers have each one element as a_i is the identity, whilst b_i permutes the two local states. In this case, therefore, the coequaliser is the terminal SGS with $L_{3,1}$ and $L_{3,2}$ being singletons and S_3 one as well. This is in agreement with the calculation for the corresponding frames after application of the reflector, $\mathcal{L}$. All the maps happen to be bounded so just from this simple example one might expect there to be a similar picture throughout $\underline{Glob.States}$, however this is far from being the case. It seems that non-observable states can cause havoc, wrecking the frame structure underlying the set of global states. This is even true in the single agent setting!

Example. Let $L_1 = \{x_1, x_2, x_3, x_4\}$, $L_2 = \{x_1, x_2, x_3\}$,

$$a(x_1) = x_2, \;\; a(x_2) = x_1, \;\; a(x_3) = x_3, \;\; a(x_4) = x_3,$$

whilst

$$b(x_1) = x_1, \;\; b(x_2) = x_2, \;\; b(x_3) = x_3, \;\; b(x_4) = x_1,$$

but $S = \{x_1, x_2, x_3\}$. Note that x_4 is not part of the set of global states although 'global' and 'local' are not good terms to use here. (We could have taken a second agent with a set of local states L_2, $S_1 = S \times L_2 \subseteq L_1 \times L_2$ and $\underline{a}, \underline{b}$ extended by the product with the identity function on L_2, however this would have obscured the fact that the problem is with the 'unobservable' states, i.e. those outside the set of global states. The problem has nothing to do with the number of agents.)

Considering S as a frame, the coequaliser of a and b (restricted to S) has two worlds, $\{x_1, x_2\}$ and $\{x_3\}$. Considered as $(S, \underline{L})$, where $\underline{L} = \{L_1\}$, the coequaliser of a and b is a singleton, since $x_1 = b(x_4)$ and $a(x_4) = x_3$. The globally unobservable x_4 has resulted in the two worlds of S/R being collapsed together.

If one now considers the essentially equivalent SGSs with L_1 replaced by L_2, the coequaliser will be as expected, i.e. S/R with local state set L_2/R, which is, of course, the same. In other words, the crude coequaliser construction in $\underline{Glob.States}$ does not respect essential equivalence. Diagrams in $\underline{Glob.States}$, which would become isomorphic if considered as diagrams in $[\underline{Glob.States}]$, can lead to coequalisers which are not essentially equivalent. The boundedness or otherwise of the morphisms involved is not the question as boundedness only involves conditions on the global states. Thus the crude coequalisers are not the answer to the problem of finding coequalisers of SGSs with bounded morphisms that correspond to those in the frames.

Remark. It might be thought that these unobservable states should be thrown out from the start in some way, e.g. by considering replete systems only, since such states are not involved with the multiagent system being modelled. The author believes that may not be the case. Increasingly, one looks for modularised information systems since they stand a better chance of being shown to be doing what they are supposed to do. A multiagent systems may consist of various components or modules and different components may involve different subsets of each agent's set of local states. To insist that all systems should be replete would hamstring any attempts to apply modular decomposition ideas in this area. This does mean that care will be needed when looking for modular decompositions in terms of colimits of interpreted systems.

We now can use, however, that $[\underline{Glob.States}]$ is equivalent to a reflexive subcategory of frames (with bounded morphisms), namely to I-$\underline{Frames}$. Suppose

$$(S_1, \underline{L}_1) \underset{\underline{b}}{\overset{\underline{a}}{\rightrightarrows}} (S_2, \underline{L}_2)$$

is a diagram in $[\underline{Glob.States}]$, so $\underline{a}$, $\underline{b}$ are determined only up to essential equivalence and we may assume both $(S_1, \underline{L}_1)$ and $(S_2, \underline{L}_2)$ are replete systems. The corresponding diagram of frames is

$$\mathcal{F}(S_1,\underline{L}_1) \underset{\mathcal{F}(\underline{b})}{\overset{\mathcal{F}(\underline{a})}{\rightrightarrows}} \mathcal{F}(S_2,\underline{L}_2)\,.$$

Now we can form a coequaliser frame, $F=(S_3,\sim_1,\ldots,\sim_n)$, where $S_3=S_2/R$ is formed as above, i.e. R is generated by

$$\underline{a}(s)\ R\ \underline{b}(s)$$

and $\sim_k$ is given by

$$[x]_R \sim_k [y]_R$$

if there are $x'\in[x]_R,\ y'\in[y]_R$ with $x'_k=y'_k$.

This frame, F need not be an I-frame, but that does not matter since we will convert it back to an SGS via the functor $\mathcal{G}$. This will give a set of global states with associated frame $F/\!\sim$ with $\sim=\bigcap\sim_k$. The individual agent's set of local states will be $S_3/\!\sim_k$ for agent k.

It would seem at first sight that this system was going to be difficult to work with since it involves a fair number of different quotienting operations for various equivalence relations. For instance, the description of a 'quotienting' morphism from $(S_2,\underline{L}_2)$ to $\mathcal{G}(F)$ initially looks difficult, but we have a description of F as a coequaliser in the diagram

$$\mathcal{F}(S_1,\underline{L}_1) \underset{\mathcal{F}(\underline{b})}{\overset{\mathcal{F}(\underline{a})}{\rightrightarrows}} \mathcal{F}(S_2,\underline{L}_2) \xrightarrow{c} F$$

and as $\mathcal{G}$ is left adjoint, it preserves coequalisers which means that

$$\mathcal{G}\mathcal{F}(S_1,\underline{L}_1) \underset{\mathcal{G}\mathcal{F}(\underline{b})}{\overset{\mathcal{G}\mathcal{F}(\underline{a})}{\rightrightarrows}} \mathcal{G}\mathcal{F}(S_2,\underline{L}_2) \xrightarrow{\mathcal{G}(c)} \mathcal{G}(F)$$

is also a coequaliser diagram. This, in fact, does the job, since we have assumed both $(S_1,\underline{L}_1)$ and $(S_2,\underline{L}_2)$ are replete so, by Proposition 1, both

$$\varepsilon_{(S_i,\underline{L}_i)}:\mathcal{G}\mathcal{F}(S_i,\underline{L}_i)\to(S_i,\underline{L}_i),\quad i=1,2,$$

are isomorphisms. As c is bounded, so is $\mathcal{G}(c)$, and we have a description of coequalisers in [$\underline{Glob.States}$].

We thus have explicit elementary constructions for colimits in both the categories, $\underline{Frames}$ and [$\underline{Glob.States}$].

Example. By way of illustration of how colimits might be used, consider a set of global states $(S,\underline{L})$ in which

$$L_1=\{l_1,\ l_2,\ l_3,\ l_4\},$$
$$L_2=\{m_1,m_2,m_3,m_4\}$$

and $S = L_1 \times L_2 \backslash \{(l_1, m_1), (l_1, m_4), (l_4, m_1), (l_4, m_4)\}$, so S is a cross shaped region. Intuitively S consists of two overlapping hypercubes. One is specified by $S_A = L_1 \times \{m_2,\ m_3\}$, the other by $S_B = \{l_2, l_3\} \times L_2$.

The overlap is also a hypercube, $S_C = \{l_2, l_3\} \times \{m_2, m_3\}$. There are two obvious maps from S_C to $S_A \sqcup S_B$ sending the overlap into each of the 'cross arms'. We leave it to the reader to check that the resulting coequaliser is the cross shaped $(S, \underline{L})$.

Models. The above handles colimits of frames and of sets of global states. How about colimits of models? There is no problem. In all cases, if the frames/SGSs are given with interpretations, $\pi_i : S_i \to \mathcal{P}(P)$, then, as the colimiting constructions are built up from colimiting operations on the underlying sets of possible worlds, these interpretations yield a cocone with vertex $\mathcal{P}(P)$ and hence, there is a unique induced interpretation on the colimit.

Proposition 11. *The categories $\underline{Comp.Models}$ and $[\underline{Int.Systems}]$ have all colimits.*

Remarks. (i) As before the problem of indistinguishable worlds having different interpreted values forces us to use $\underline{Comp.Models}$ not $\underline{Kripke.Models}$.

(ii) Of course, the above proposition is also a consequence of the general coalgebraic approach to modal logics. This paper only scratches the surface of the potential uses of that general theory to MASs. (For more on coalgebras and their link with modal logics, see, for instance the excellent introductory notes by Kurz [11], and the references therein.)

6. Limits?

Dual to coproducts, one has the question of products of both models/frames and interpreted systems/SGSs. In general modal logic, the intuitive constructions do not always work. The standard example is with F_1, an irreflexive frame on a single world and F_2, a reflexive one again on one world, then the obvious candidate, $F_1 \times F_2$ does not give the product frame. The problem can be quite subtle, but need not concern us here as the logics involved are not directly applicable to multiagent systems.

The question of products will be the subject of a later paper in this project as it requires the development of some different techniques including a discussion of polymonadic algebras as models for multiagent systems and the extended Stone duality between these and generalised frames. It can also be approached by coalgebraic methods.

The reason for wanting a development of limits of models/frames and interpreted systems/SGSs is the potential of these structures for detailed analyses of 'knowledge update'. Initial investigations show that pullbacks (a simple form of limit) seem to give a powerful descriptive tool for studying update in MAS. Mathematically, existing descriptions of knowledge and belief update strongly resemble certain well known constructions from topology and recent links between topological ideas and both distributed systems and modal logic proof systems seem to indicate that this relationship is potentially important in understanding the evolution of knowledge in a multiagent setting.

7. Conclusion and future directions

The categorical comparison between Kripke frames and sets of Global States extends the results of Lomuscio, van der Meyden and Ryan, but in a non-trivial way. The equivalences of categories that have been exhibited emphasise aspects of these models for multiagent systems that were previously not taken into account. In particular the existence of non-observable local states in an SGS or interpreted system has consequences when considering colimits of such systems, whilst the existence of indistinguishable worlds in a Kripke frame, can cause problems of compatibility with the interpretation.

The categorical analysis of these models for MASs suggests new directions, in particular decomposition theorems yielding a potential for 'modularisation' of multiagent settings. This, and the knowledge update problem mentioned in the last section, would seem to need a thorough treatment of the algebraic semantics, [6], of the modal logics used for MASs. Given that algebraic models for MASs could potentially be analysed by the powerful mathematical tools available within several branches of algebra, some of this would seem worth investigating as a step towards a greater mathematisation of the analysis of multiagent systems so as to complement the logical analysis that has been so successfully applied up to now. This is the aim of the project 'Categorical Perspectives on Multiagent Systems' mentioned earlier. Another aspect of this project can be seen in [17].

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